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ELECTRIC AND THERMAL BEHAVIOR OF COMPLETELY IONIZED DEUTERIUM IN PRESENCE OF A MAGNETIC FIELD

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ABSTRACT

Completely ionized deuterium has a large thermal conductivity

$$K_c = 6.3 \times 10^{27}\ T_{(Kev)}^{5/2}\ cm^{-1}\ sec^{-1}\ k$$

(k is the Boltzmann constant). By applying a magnetic field H at right angles to the temperature gradient the conductivity reduces to K_H. In Fig. 5 the ratio K_H/K_o is plotted against H/H_c where $H_c = 2120\ T_{(Kev)}^{-3/2}$ kilogauss. This beneficial effect will disappear quickly because the magnetic lines of force tend to be squeezed out of the region of high temperature because of electric currents arising perpendicular to both the magnetic field and the temperature gradient. In a slab of thickness 2a (cm), which has a temperature T_o at the center, H decreases at a rate

$$dH/dt = -46.6 \left[T_{o(Kev)}/a^2\right] \text{ kilogauss}/\mu\text{sec}$$

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ELECTRIC AND THERMAL BEHAVIOR OF COMPLETELY IONIZED DEUTERIUM IN PRESENCE OF A MAGNETIC FIELD

Introduction

Completely ionized deuterium has a large thermal conductivity, which can be reduced by applying a magnetic field at right angles to the temperature gradient. There is a tendency for the magnetic lines of force to be squeezed out of the region of high temperature because of electric currents arising perpendicularly to both the magnetic field and the temperature gradient. It is of practical interest to know by how much the conductivity can be reduced and for how long a time the magnetic field can be kept high enough to keep the reduction effective. For this purpose we derive in Part I of this report general expressions for the "conductivity coefficients" which relate the electric and thermal currents to the electric field and the temperature gradient in the presence of a magnetic field. In Part II, we discuss the electric and magnetic field in the heated deuterium by entering the expressions for the electric currents derived in Part I into Maxwell's equations. We can thus determine the effective heat conductivity and the rate of decay of the magnetic field.

PART I -- Kinetic Theory

Let us assume a magnetic field H in the y - direction and an electric field and a temperature gradient both in the x - z plane. There will in general result electric and heat currents in the x and z directions.

$$
\begin{aligned}
j_x &= \sigma_{\parallel} E_x - \sigma_{\perp} E_z + \tau_{\parallel}\, \partial T/\partial x - \tau_{\perp}\, \partial T/\partial z \\
j_z &= \sigma_{\perp} E_x + \sigma_{\parallel} E_z + \tau_{\perp}\, \partial T/\partial x + \tau_{\parallel}\, \partial T/\partial z \\
q_x &= -\mu_{\parallel} E_x + \mu_{\perp} E_z - K_{\parallel}\, \partial T/\partial x + K_{\perp}\, \partial T/\partial z \\
q_z &= -\mu_{\perp} E_x - \mu_{\parallel} E_z - K_{\perp}\, \partial T/\partial x - K_{\parallel}\, \partial T/\partial z
\end{aligned}
\qquad (1)
$$

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In the presence of a pressure gradient $\underline{E}$ has to be replaced by $\underline{E}+(kT/-e)(1/p)\nabla p$, where $-e = 4.8 \times 10^{-10}$ esu. These expressions will be obtained by extending the theory of LA-334 in the following manner:

1) for the electron distribution function we try the form:

$$\phi(x,z,\underline{v}) = f(x,z,v)\left[1+v_x\, h_x(v)+v_z\, h_z(v)\right] \tag{2}$$

2) In the Boltzmann equation, we include the magnetic field by using:

$$D(\phi) = v_x \frac{\partial\phi}{\partial x} + v_z \frac{\partial\phi}{\partial z} + \frac{e}{m}\left[(E_x - \frac{v_z}{c} H)\frac{\partial\phi}{\partial v_x} + (E_z + \frac{v_x}{c} H)\frac{\partial\phi}{\partial v_z}\right] \tag{3}$$

It is no longer possible to completely replace $D(\phi)$ by $D(f)$. If that were done, H would completely drop out of the equation. Instead one has to keep ϕ in those parts of D which contain H. This is a consistent scheme inasmuch as it keeps just the terms which are linear in v_x and v_z in the equation. The significance of this approximation is, that electric fields and the gradients of pressure and temperature are considered as being small, whereas the magnetic field may have any value. Let us set: $\epsilon = mv^2/2kT$, $\omega = -eH/mc$ and put $D(\phi)$ into the form

$$\begin{aligned} D(\phi) = &\left(\frac{-eE_x}{kT} L_0(\epsilon) - \frac{1}{T}\frac{\partial T}{\partial x} L_1(\epsilon) - \omega h_z\right) v_x f \\ &+ \left(\frac{-eE_z}{kT} L_0(\epsilon) - \frac{1}{T}\frac{\partial T}{\partial z} L_1(\epsilon) + \omega h_x\right) v_z f \end{aligned} \tag{4}$$

(in presence of a pressure gradient its force must be added to that of the electric field). The $L_r(\epsilon)$ are Laguerre polynomials of order 3/2. We now expand:

$$h_x = \Sigma p_s\, L_s(\epsilon), \quad h_z = \Sigma q_s\, L_s(\epsilon) \tag{5}$$

and set

$$\begin{aligned} A_x &= eE_x/kT, & B_x &= (1/T)(\partial T/\partial x) \\ A_z &= eE_z/kT, & B_z &= (1/T)(\partial T/\partial z) \end{aligned} \tag{6}$$

and obtain:

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$$-D(\phi) = \left[A_x\, L_0(\epsilon) + B_x\, L_1(\epsilon) + \omega \sum q_s\, L_s(\epsilon)\right] V_x f + \left[A_z\, L_0(\epsilon) + B_z\, L_1(\epsilon) - \omega \sum p_s\, L_s(\epsilon)\right] V_z f \qquad (7)$$

On the right-hand side of the Boltzmann equation we must replace $V_x h(v)$ as it was used in Eqs. (46) and (56) of LA-334 by $V_x h_x(v)$ and then by $V_z h_z(v)$ and add the expression obtained by these operations. We now multiply the Boltzmann equation on both sides in turn by $(m/nkT)\, v_x\, L_r(\epsilon)$ and $(m/nkT)\, v_z\, L_r(\epsilon)$ and integrate over $d\underline{v}$. This procedure yields the two equations:

$$\left.\begin{aligned} A_x\, \delta_{or} + (5/2)\, B_x\, \delta_{1r} + \frac{\Gamma(r+5/2)}{\Gamma(r+1)\,\Gamma(5/2)}\, \omega q_r &= \sum p_s\, H_{rs} \\ A_z\, \delta_{or} + (5/2)\, B_z\, \delta_{1r} - \frac{\Gamma(r+5/2)}{\Gamma(r+1)\,\Gamma(5/2)}\, \omega p_r &= \sum q_s\, H_{rs} \end{aligned}\right. \qquad (8)$$

The matrix elements on the right hand side are defined by

$$H_{rs} = \nu\, h_{rs} \qquad (9)$$

where

$$\nu = (4\sqrt{2\pi}/3)\, \lambda n\, (e^2/kT)^2\, (kT/m)^{1/2} \qquad (10)$$

using λ, h^e_{rs}, h^i_{rs} as defined in Eqs. (46), (75), and (78) of LA-334. ν^{-1} can essentially be interpreted as the time between collisions of the electron. It seems worth while to warn the reader that the H_{rs} as used in this report differ from those used in LA-334 by a factor m/nkT.

The special symmetry of problems in which a magnetic field is involved and which reflects itself both in Eqs. (1) and (8) suggests the introduction of complex quantities. Let us define

$$\left.\begin{aligned} j &= j_x + i\, j_z, & q &= q_x + i\, q_z \\ E &= E_x + i\, E_z & \nabla T &= (\partial T/\partial x) + i\, (\partial T/\partial z) \\ \sigma &= \sigma_{\parallel} + \sigma_{\perp} & \tau &= \tau_{\parallel} + i\, \tau_{\perp} \end{aligned}\right\} \qquad (12)$$

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$$\left.\begin{array}{ll} \mu = \mu_{\parallel} + i\,\mu_{\perp} & \kappa = \kappa_{\parallel} + i\,\kappa_{\perp} \\ A = A_x + i\,A_z & B = B_x + i\,B_z \end{array}\right\} \tag{12}$$

$$C_r = p_r + i\,q_r$$

Then we can rewrite Eqs. (1) and (8) in much simpler forms, namely:

$$j = \sigma E + \tau \nabla T; \quad q = -\mu E - \kappa \nabla T \tag{1a}$$

and

$$A\,\delta_{or} + (5/2)\,B\,\delta_{1r} - \frac{\Gamma(r+5/2)}{\Gamma(r+1)\,\Gamma(5/2)}\,i\,\omega C_r = \nabla \Sigma C_s\,h_{rs} \tag{8a}$$

More explicitly we can write:

$$\left\{\begin{array}{llll} \nu h_{oo} + i\omega, & \nu h_{o1}, & \nu h_{o2} & \cdots \\ \nu h_{o1}, & \nu h_{11} + (5/2)\,i\omega, & \nu h_{12} & \cdots \\ \nu h_{o2}, & \nu h_{12}, & \nu h_{22} + (35/8)\,i\omega, & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{array}\right\} \left\{\begin{array}{c} C_o \\ C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}\right\} = \left\{\begin{array}{c} A \\ (5/2)\,B \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}\right\} \tag{13}$$

We want to solve this infinite system of equations for C_o and C_1, because these two coefficients yield j and q (see Eqs. (15) and (16) of LA-334). In order to do this we shall use only that part of the matrix actually written down in (13). This approximation was shown to be sufficiently accurate in absence of the magnetic field and we shall demonstrate its validity in the general case by checking the method for the case of negligible e - e scattering.

We can then write down immediately the solution of (13):

$$\begin{aligned} C_o &= \nu^{-1}\left[(\Delta_{oo}/\Delta)\,A - (5/2)\,(\Delta_{o1}/\Delta)\,B\right] \\ C_1 &= \nu^{-1}\left[-(\Delta_{o1}/\Delta)\,A + (5/2)\,(\Delta_{11}/\Delta)\,B\right] \end{aligned} \tag{14}$$

where

$$\Delta = \left|\begin{array}{lll} (h_{oo} + i\,\omega/\nu), & h_{o1}, & h_{o2} \\ h_{o1}, & \left[h_{11} + (5/2)\,i\,\omega/\nu\right], & h_{12} \\ h_{o2}, & h_{12} & \left[h_{22} + (35/8)\,i\,\omega/\nu\right] \end{array}\right| \tag{15}$$

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and Δ_{oo}, Δ_{o1}, Δ_{11} are minors of the determinant Δ. From (14) we derive further that:

$$\sigma = \frac{ne^2}{m\nu}\frac{\Delta_{oo}}{\Delta}, \qquad \tau = \frac{(-e)nk}{m\nu}\frac{5}{2}\frac{\Delta_{o1}}{\Delta}$$

$$\mu = \frac{(-e)nkT}{m\nu}\frac{5}{2}\left(\frac{\Delta_{oo}+\Delta_{o1}}{\Delta}\right), \qquad K = \frac{nk^2T}{m\nu}\frac{25}{4}\left(\frac{\Delta_{o1}+\Delta_{11}}{\Delta}\right) \tag{16}$$

In Part II we shall see that the effective heat conductivity is obtained by giving E that value for which $j = 0$, namely $E = -(\tau/\sigma)\Delta T$. This leads to

$$q = -K_{eff}\nabla T \tag{17}$$

with

$$\begin{aligned} K_{eff} &= K - \tau\mu/\sigma \\ &= \frac{25}{4}\frac{nk^2T}{m\nu}\left(\Delta_{11}\ \Delta_{oo} - \Delta_{o1}^2\right) \end{aligned} \tag{18}$$

This can be transformed just as it was done in Eq. (24) of LA-334 by applying Sylvester's theorem, and leads to

$$K_{eff} = \frac{25}{4}\frac{nk^2T}{m\nu}\frac{h_{22}+(35/8)\ i(\omega/\nu)}{\Delta_{oo}}$$

In particular we want the real part of this expression which represents the conductivity in direction of the temperature gradient. We will denote this real part by K_H. Numerical calculations were carried out for $f_{\sigma_{\parallel}}$ $if_{\sigma_{\perp}} = \Delta_{oo}/\Delta$, Δ_{o1}/Δ, and Δ_{11}/Δ as well as for the derived quantities

$$f_{\tau_{\parallel}} + if_{\tau_{\perp}} = (5/2)(\Delta_{o1}/\Delta), \qquad f_{\mu_{\parallel}} + if_{\mu_{\perp}} = (5/2)\ (\Delta_{oo}+\Delta_{o1})/\Delta$$

$$f_{K_{\parallel}} + if_{K_{\perp}} = (25/4)\ (\Delta_{o1}+\Delta_{11})/\Delta$$

and

$$f_{K_H} = \text{real part of } (25/4)\left[\frac{h_{22}+(35/8)\ i(\omega/\nu)}{\Delta_{oo}}\right].$$

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TABLE I

ω/ν	$f_{\sigma_\parallel} + i f_{\sigma_\perp} = \Delta_{00}/\Delta$	Δ_{01}/Δ	Δ_{11}/Δ
0	1.950	.5546	.6636
.2	1.558 - .7290 i	.3783 - .2834 i	.5278 - .2563 i
.5	.8601 - .9186 i	.1040 - .2713 i	.2737 - .3027 i
1.0	.3603 - .7025 i	-.00809 - .1454 i	.1208 - .2201 i
2.0	.1246 - .4192 i	-.02389 - .06047 i	.05364 - .1328 i
4.0	.04274 - .2287 i	-.01703 - .01926 i	.02507 - .07933 i
6.0	.02226 - .1581 i	-.01089 - .008093 i	.01467 - .05803 i
asympt.	$(\omega/\nu)^{-2} - (\omega/\nu)^{-1} i$	$-.6(\omega/\nu)^{-2} - 2.64(\omega/\nu)^{-3} i$	$.746(\omega/\nu)^{-2} - .4(\omega/\nu)^{-1} i$

ω/ν	$f_{\tau_\parallel}$	$f_{\tau_\perp}$	$f_{\mu_\parallel}$	$f_{\mu_\perp}$	$f_{\kappa_\parallel}$	$f_{\kappa_\perp}$	f_{κ_H}
0	1.386	0	6.261	0	7.614	0	3.1614
.2	.9458	-.7085	4.841	-2.531	5.663	-3.373	2.7742
.5	.2600	-.6782	2.410	-2.975	2.361	-3.588	1.7374
1.0	-.0202	-.3635	.8805	-2.120	.7044	-2.284	.8477
2.0	-.0597	-.1512	.2518	-1.199	.1859	-1.208	.3871
4.0	-.0426	-.0481	.0643	--.6199	.0502	-.6162	.1739
6.0	-.0272	-.0202	.0284	-.4176	.0236	-.4133	.09806
asympt.	$-1.5(\frac{\omega}{\nu})^{-2}$	$-6.6(\frac{\omega}{\nu})^{-3}$	$(\frac{\omega}{\nu})^{-2}$	$-2.5(\frac{\omega}{\nu})^{-1}$	$.913(\frac{\omega}{\nu})^{-2}$	$-2.5(\frac{\omega}{\nu})^{-1}$	$4.664(\frac{\omega}{\nu})^{-2}$

In the following section we check the validity of our approximation by treating the electron scattering as negligible. The Boltzmann equation simplifies to:

$$D(\emptyset) = -4\pi\lambda n(e^2/mv^2)^2 \, V(V_x \, h_x + v_z \, h_z) \, f \qquad (19)$$

(See Eq. (56) of LA-334).

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Introducing ν and ϵ and dividing by f this can be written as

$$\left[A_x L_o(\epsilon) + B_x L_1(\epsilon) + \omega h_z\right] V_x + \left[A_z L_o(\epsilon) + B_z L_1(\epsilon) - \omega h_x\right] V_z = (3\sqrt{\pi}/4)\ \nu\epsilon^{-3/2}(V_x h_x + V_z h_z) \tag{20}$$

This splits into the two equations for the factors of V_x and V_z which can be re-combined by the use of complex notation

$$A\ L_o(\epsilon) + B\ L_1(\epsilon) - i\omega h(\epsilon) = (3\sqrt{\pi}/4)\ \nu\epsilon^{-3/2}\ h(\epsilon) \tag{21}$$

where

$$h = h_x + i\ h_z \tag{22}$$

This is just an algebraic equation for h and leads at once to

$$h(\epsilon) = \frac{A\ L_o(\epsilon) + B\ L_1(\epsilon)}{(3\sqrt{\pi}/4)\ \nu\epsilon^{-3/2} + i\omega} \tag{23}$$

The electric and thermal current can be found by using the expressions

$$j = (e/3)\int v^2\ hfd\underline{v} \quad \text{and} \quad q = (m/6)\int v^4\ hfd\underline{v} \tag{24}$$

Now

$$fd\underline{v} = 2n\,\pi^{-\frac{1}{2}}\ e^{-\epsilon}\ \epsilon^{\frac{1}{2}}\ d\epsilon \tag{25}$$

So that we have

$$j = \frac{4}{3\sqrt{\pi}}\frac{nekT}{m}\int_0^\infty \epsilon^{3/2}\ e^{-\epsilon}\ h(\epsilon)\ d\epsilon$$

$$q = \frac{4}{3\sqrt{\pi}}\frac{n(kT)^2}{m}\int_0^\infty \epsilon^{5/2}\ e^{-\epsilon}\ h(\epsilon)\ d\epsilon \tag{26}$$

In order to carry out these integrations we set

$$\xi = \left(\frac{3\sqrt{\pi}}{4}\frac{\nu}{\omega}\right)^{2/3} \tag{27}$$

so that

$$h = \omega^{-1}\frac{\epsilon^{3/2}\left[A\ L_o(\epsilon) + B\ L_1(\epsilon)\right]}{\xi^{3/2} + i\epsilon^{3/2}}$$

$$= \omega^{-1}\frac{\left[\xi^{3/2}\epsilon^{3/2} + i\ \epsilon^3\right]\left[A + B(5/2 - \epsilon)\right]}{\xi^3 + \epsilon^3} \tag{28}$$

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We now introduce the integrals:

$$I_n(\xi) = \int_0^\infty \frac{\epsilon^n e^{-\epsilon}}{\xi^3+\epsilon^3}\, d\epsilon \tag{29}$$

and express j and q in terms of these integrals.

$$j = \frac{4}{3\sqrt{\pi}} \frac{nekT}{n\omega} \left[(A+\frac{5}{2}B)(\xi^{3/2} I_3 - iI_{4.5}) - B(\xi^{3/2} I_4 - iI_{5.5}) \right]$$

$$q = \frac{4}{3\sqrt{\pi}} \frac{n(kT)^2}{m\omega} \left[(A+\frac{5}{2}B)(\xi^{3/2} I_4 - iI_{5.5}) - B(\xi^{3/2} I_5 - iI_{6.5}) \right] \tag{30}$$

This leads at once to the relations

$$f_{\sigma_\parallel} = \frac{4}{3\sqrt{\pi}} \frac{\nu}{\omega} \xi^{3/2} I_3 = \left(\frac{\nu}{\omega}\right)^2 I_3$$

$$f_{\sigma_\perp} = - \frac{4}{3\sqrt{\pi}} \frac{\nu}{\omega} I_{4.5}$$

$$f_{\tau_\parallel} = \left(\frac{\nu}{\omega}\right)^2 (I_4 - \frac{5}{2} I_3)$$

$$f_{\tau_\perp} = \frac{4}{3\sqrt{\pi}} \left(\frac{\nu}{\omega}\right)^2 (\frac{5}{2} I_{4.5} - I_{5.5})$$

$$f_{\mu_\parallel} = \left(\frac{\nu}{\omega}\right)^2 I_4 \tag{31}$$

$$f_{\mu_\perp} = - \frac{4}{3\sqrt{\pi}} \left(\frac{\nu}{\omega}\right) I_{5.5}$$

$$f_{\kappa_\parallel} = \left(\frac{\nu}{\omega}\right)^2 (I_5 - \frac{5}{2} I_4)$$

$$f_{\kappa_\perp} = \frac{4}{3\sqrt{\pi}} \left(\frac{\nu}{\omega}\right) (\frac{5}{2} I_{5.5} - I_{6.5})$$

which permit us to check the f values found by the matrix method. The integrals I_n are evaluated in the appendix. We notice in both theories the relationship

$$f_\mu = (5/2)\, f_\sigma + f_\tau \tag{32}$$

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therefore we do not have to check the components of one of them, say $f_{\tau_\parallel}$ and $f_{\tau_\perp}$. Figs. 1 and 2 show that the agreement is very good for $f_{\sigma_\parallel}$, $f_{\sigma_\perp}$, $f_{\mu_\parallel}$, and $f_{\mu_\perp}$. For $f_{\kappa_\parallel}$ (Fig. 3) which changes sign, the agreement is not very good. Of particular interest is of course $f_{\kappa_\parallel}$ (Fig. 4) for which the agreement is fair. In all diagrams the full-line curves correspond to the matrix theory and the broken-line curves to the differential-equation theory.

PART II

Let us consider a slab of deuterium parallel to the y - z plane which is placed in a homogeneous magnetic field in the y direction. This slab is suddenly heated and subsequently maintained at a temperature, which we assume to be a known function $T(x)$, and which is of the order of magnitude of several Kev. This situation will, of course, give rise to electric and thermal currents in the x and z direction. The electric current in the x direction dies out very rapidly as we can see from the following argument. Let us call the current due to temperature and pressure gradients j_{xo}. The current will then establish an electric field E_x by the relation:

$$\partial E_x/\partial t + 4\pi j_x = 0$$

($\partial H/\partial z = 0$, because of the symmetry of the problem) where

$$j_x = \sigma_\parallel E_x + j_{xo}$$

Thus we have

$$\partial E_x/\partial x + 4\pi(j_{ox} + \sigma_\parallel E_x) = 0$$

which means that E_x will approach a value for which $j_x = 0$ during a time of the order of magnitude $1/4\pi\sigma_\parallel$. By Eqs. (10) and (16) and using $\lambda = 9.67$ as computed from Eq. (46) of LA-334 for a temperature of around 30 Kev and a density of 4×10^{22} electrons/cm^3 we find $\sigma_\parallel = 3.3 \times 10^{21}(kT/mc^2)^{3/2} f_{\sigma_\parallel}$ sec^{-1}. The time of adjustment

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thus turns out to be $2.5 \times 10^{-23}(kT/mc^2)^{-3/2}\, f_{\sigma_{\parallel}}^{-1}$ sec, which is an exceedingly short time. We may therefore say that the electric field E_x immediately counteracts any effects which tend to cause a current in the x-direction. That means that we can set the first of the Eqs. (1) equal to zero and use this relation to eliminate E_x from the other ones. Applying this procedure for the second equation we obtain:

$$j_z = \sigma E_z - \tau\, dT/dx \tag{33}$$

if we set

$$\sigma = \sigma_{\parallel} + \sigma_{\perp}^2/\sigma_{\parallel} \quad \text{and} \quad \tau = \sigma_{\perp}\tau_{\parallel}/\sigma_{\parallel} - \tau_{\perp} \tag{34}$$

($\partial T/\partial z = 0$ by assumption).

The sudden heating will therefore give rise to a current:

$$j_{zo} = - \tau\, dT/dx \tag{35}$$

which will be established in a time of the order of magnitude of the time between atomic collisions that is about 10^{-11} sec. This current tends to push the magnetic lines of force out, which induces a field E_z, which in turn slows down the process of pushing out the field. In order to investigate this sequence let us solve

$$\partial H/\partial x = (4\pi/c)(j_{zo} + \sigma E_z) + (1/c)\, \partial E_z/\partial t \tag{36}$$

$$\partial E_z/\partial x = (1/c)\, \partial H/\partial t \tag{37}$$

subject to the condition that at $t = 0$, $\partial H/\partial x = 0$ and $E_z = 0$. The extremely large value of σ inside the slab will make the term $(1/c)\, \partial E_z/\partial t$ effectively drop out of the equation and of course remove the condition $E_z = 0$. In order to show this we must show that during the time that E_z is "switched on" $\partial H/\partial x$ remains sufficiently small so that it can still be regarded as zero. Let us for the moment make this assumption. We can then solve (36) and obtain (regarding σ as constant)

$$E_z = - (1/\sigma)\, j_{zo}\, (1 - e^{-4\pi\sigma t})$$

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The transient term disappears when, let us say, $4\pi\sigma t_1 = 10$. From Eq. (37) we find

$$\frac{\partial}{\partial t}\frac{\partial H}{\partial x} = c\,\frac{\partial^2 E_z}{\partial x^2} = -\frac{c}{\sigma}\, j_{zo}''\,(1 - e^{-4\pi\sigma t})$$

and

$$\frac{\partial H}{\partial x} = -\frac{c}{\sigma}\, j_{zo}''\left(t - \frac{1 - e^{-4\pi\sigma t}}{4\pi\sigma}\right)$$

At $t = t_1$

$$\frac{\partial H}{\partial x} = -\frac{c}{\sigma}\, j_{zo}''\,\frac{g}{4\pi\sigma}$$

This should be compared to $(4\pi/c)j_{zo}$ and is found to be smaller by a factor $g(c/4\pi\sigma)^2\, j_{zo}''/j_{zo}$ which is of the order of magnitude 10^{-20} or less. This shows that our assumption is self consistent. During the switching on period $j_z = j_{zo} + \sigma E$ drops to zero.

This is of importance for the calculation of the initial value of the effective heat conductivity. It means that we can set $j = j_x + i j_z = 0$ and eliminate E from (1a) as it was done in Eqs. (17) and (18). By taking the real part of κ_{eff} we obtain the conductivity in direction of the temperature gradient,

$$\kappa_H = (nk^2T/m\nu)\, f_{\kappa_H} \tag{38}$$

where f_{κ_H} is tabulated in Table I.

In the case of liquid deuterium with a density of $n = 4.2 \times 10^{22}$ cm^{-3} the critical value of the magnetic field at which $\omega = \nu$ is

$$H_c = 2120\; T_{(Kev)}^{-3/2} \text{ kilogauss} \tag{39}$$

In the absence of a magnetic field the thermal conductivity is

$$\kappa_o = 6.3 \times 10^{27}\; T_{Kev}^{5/2}\ \text{cm}^{-1}\ \text{sec}^{-1} \cdot k \tag{40}$$

(k = Boltzmann constant). In Fig. 5 the ratio κ_H/κ_o is plotted against H/H_c. We see from this graph that one can obtain a considerable reduction of heat losses by means of a sufficiently strong magnetic field.

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If j_z is not taken to be zero the heat current can be expressed by the equation

$$q_x = -K_H \frac{dT}{dx} + \frac{\sigma_{\parallel}\mu_{\perp} - \sigma_{\perp}\mu_{\parallel}}{\sigma_{\parallel}^2 + \sigma_{\perp}^2}\, j_z$$

By inspection of Table I we see that the factor of j_z is negative. Now j_z tends to have the opposite sign as dT/dx so that the gradual appearance of j_z tends to decrease the thermal flow. This beneficial effect, however, is insignificant because as the current j_z appears, the magnetic lines of force are squeezed out of the deuterium.

It is a critical question if the field can be maintained for a sufficiently long time to be effective. Since the rate at which the field decays has the largest value right at the beginning, we can answer this question by calculating this initial rate of decay. From Maxwell's equations (e.g., 37) we deduce that:

$$\partial H/\partial t = c\ (\partial E_z/\partial x)$$

but we have seen that initially

$$E_z = -(1/\sigma)\ j_{z0} = (\tau/\sigma)(dT/dx) \tag{42}$$

so that

$$\frac{\partial H}{\partial t} = c\,\frac{\partial}{\partial x}\left(\frac{\tau}{\sigma}\frac{dT}{dx}\right) \tag{43}$$

At the center of the slab $dT/dx = 0$ so that

$$\partial H/\partial t = c(\tau/\sigma)(d^2T/dx^2) \tag{44}$$

We now introduce

$$\frac{\tau}{\sigma} = \frac{\tau_{\parallel}\sigma_{\perp} - \tau_{\perp}\sigma_{\parallel}}{\sigma_{\parallel}^2 + \sigma_{\perp}^2} = \frac{k}{-e} f_{\tau/\sigma} \tag{45}$$

where the expression

$$f_{\tau/\sigma} = \frac{f_{\tau_{\parallel}} f_{\sigma_{\perp}} - f_{\tau_{\perp}} f_{\sigma_{\parallel}}}{f_{\sigma_{\parallel}}^2 + f_{\sigma_{\perp}}^2} \tag{46}$$

[redacted]

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 [redacted]

has been computed and represented graphically as a function of H/H_c in Fig. 6. If we assume T to be of the form $T = T_o\left[1 - (x/a)^2\right]$ we obtain:

$$\frac{\partial H}{\partial t} = -2\,\frac{ckT_o}{-ea^2}\,f_{\tau/\sigma} \tag{47}$$

We see from the graph of $f_{\tau/\sigma}$ that it does not show much variation with H/H_c in the important region, $H/H_c > 1$, and we shall use the maximum value $f_{\tau/\sigma} = .233$. This yields:

$$\frac{\partial H}{\partial t} = -46.6\,\frac{T_o\ \text{Kev}}{a^2}\,\frac{\text{Kilogauss}}{\mu\text{sec}} \tag{48}$$

One can easily show that in case of a cylinder of radius a one has to multiply this result by 2.

Up to now, we have not included any hydrodynamics in the problem. But it is clear that the high pressure inside the material will bring about an expansion. In absence of a magnetic field the kinetic theory is completely independent of the material velocity. If a magnetic field is present, however, this is no longer the case. If we designate the material velocity in the x direction with u we find that E_z has to be replaced by $E_z + (u/c)H$. Let us disregard for the discussion of this effect the current j_{zo} whose importance in producing a decay of the magnetic field was discussed in the previous paragraph. We have the equations

$$\frac{\partial H}{\partial x} = \frac{4\pi\sigma}{c}\left(E + \frac{u}{c}H\right) \tag{49}$$

$$\frac{\partial E}{\partial x} = \frac{1}{c}\frac{\partial H}{\partial t} \tag{50}$$

We combine them to

$$\frac{\partial}{\partial x}\left(\frac{c^2}{4\pi\sigma}\frac{\partial H}{\partial x}\right) = \frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(uH) \tag{51}$$

The left hand side of this equation would be of importance only if our time scale was of the order of magnitude of $a^2\sigma/c^2$ which is obviously not the case. We can

[redacted]

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therefore set

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(uH) = 0 \tag{52}$$

In the material coordinate system we can write this as

$$dH/dt = - H\,\partial u/\partial x \tag{53}$$

At the same time we have the equation of continuity

$$d\rho/dt = - \rho\,\partial u/\partial x \tag{54}$$

and therefore

$$\frac{1}{H}\frac{dH}{dt} = \frac{1}{\rho}\frac{d\rho}{dt} \tag{55}$$

Integrating this we get

$$H/H_o = \rho/\rho_o \tag{56}$$

which means that the lines of force are carried along with the material. At the same time we know from Eq. (10) that ν and therefore the critical field strength H_c are proportional to the density, so that the important ratio H/H_c is unaffected.

APPENDIX

The integrals $I_n(x) = \int_0^\infty \frac{Y^n\, e^{-Y}}{x^3 + Y^3}\, dY$ obey the recursion formula

$$I_{n+3} = n! - x^3 I_n \tag{57}$$

For $n = (0, 1, 2)$ they can be evaluated by the following method. Set

$$\epsilon = \tfrac{1}{2}(-1 + i\sqrt{3}) = e^{2\pi i/3}$$

then:

$$\frac{3x^{2-n}\, Y^n}{Y^3 + x^3} = \frac{a_n}{Y+x} + \frac{b_n}{Y+\epsilon x} + \frac{\bar{b}_n}{Y+\bar{\epsilon} x}$$

with $a_n = (1, -1, 1)$, $b_n = (\epsilon, -\bar{\epsilon}, 1)$. We obtain

$$3x^{2-n}\, I_n(x) = a_n \int_0^\infty \frac{e^{-Y}}{Y+\epsilon}\, dy + 2\mathrm{Re}\left(b_n \int_0^\infty \frac{e^{-Y}}{Y+\epsilon x}\, dY\right) \tag{58}$$

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(Re stands for real part of). The first integral is simply

$$\int_0^\infty \frac{e^{-Y}}{Y+x}\,dY = e^x E_1(x)$$

The second integral can be transformed to:

$$\int_0^\infty \frac{e^{-Y}}{Y+\epsilon x}\,dY = e^{\epsilon x}\int_{\epsilon x}^\infty \frac{e^{-t}}{t}\,dt = e^{\epsilon x}\left(\int_{\epsilon x}^{x} \frac{e^{-t}}{t}\,dt + E_1(x)\right) \tag{59}$$

The integral from ϵx to x can be carried along the circular arc and if we set $t = xe^{i\phi}$, $dt/t = i\,d\phi$ we obtain:

$$\int_{\epsilon x}^{x} \frac{e^{-t}}{t}\,dt = i\int_{2\pi/3}^{0} e^{-xe^{i\phi}}\,d\phi = -\frac{2\pi i}{3} - (1-\epsilon)\,\alpha(x) + (1-\bar{\epsilon})\,\beta(x) \tag{60}$$

where

$$\alpha(x) = x - x^4/4.4! + x^7/7.7! - + \ldots \tag{61}$$

$$\beta(x) = x^2/2.2! - x^5/5.5! + x^8/8.8! - + \ldots \tag{62}$$

are two very rapidly converging series obtained by integrating the power expansion of $e^{-xe^{i\phi}}$. We can now enter the integrals (59) and (60) into (58) and obtain I_0, I_1, and I_2 and from those by means of (57) also the I_n for higher integral values of n. For half-integral values of n we obtained the I_n by interpolation. The results of these calculations are collected in Table II.

TABLE II

x	I_3	I_4	$I_{4.5}$	I_5	$I_{5.5}$
0	1.0000	1.0000	1.3293	2.0000	3.3233
.5	.5854	.8702	1.2366	1.9234	3.2516
1	.3717	.6882	1.0557	1.7201	3.0269
2	.1725	.4130	.6976	1.2454	2.3448
3	.09070	.2523	.4564	.8643	1.7188
4	.05214	.1599	.3024	.5992	1.2372
5	.03211	.1053	.2059	.4209	.8969
6	.02088	.07175	.1437	.3012	.6565
large X	$\frac{6}{X^3+120}$	$\frac{24}{X^3+210}$	$\frac{52.344}{X^3+268.125}$	$\frac{120}{X^3+336}$	$\frac{287.892}{X^3+414.375}$

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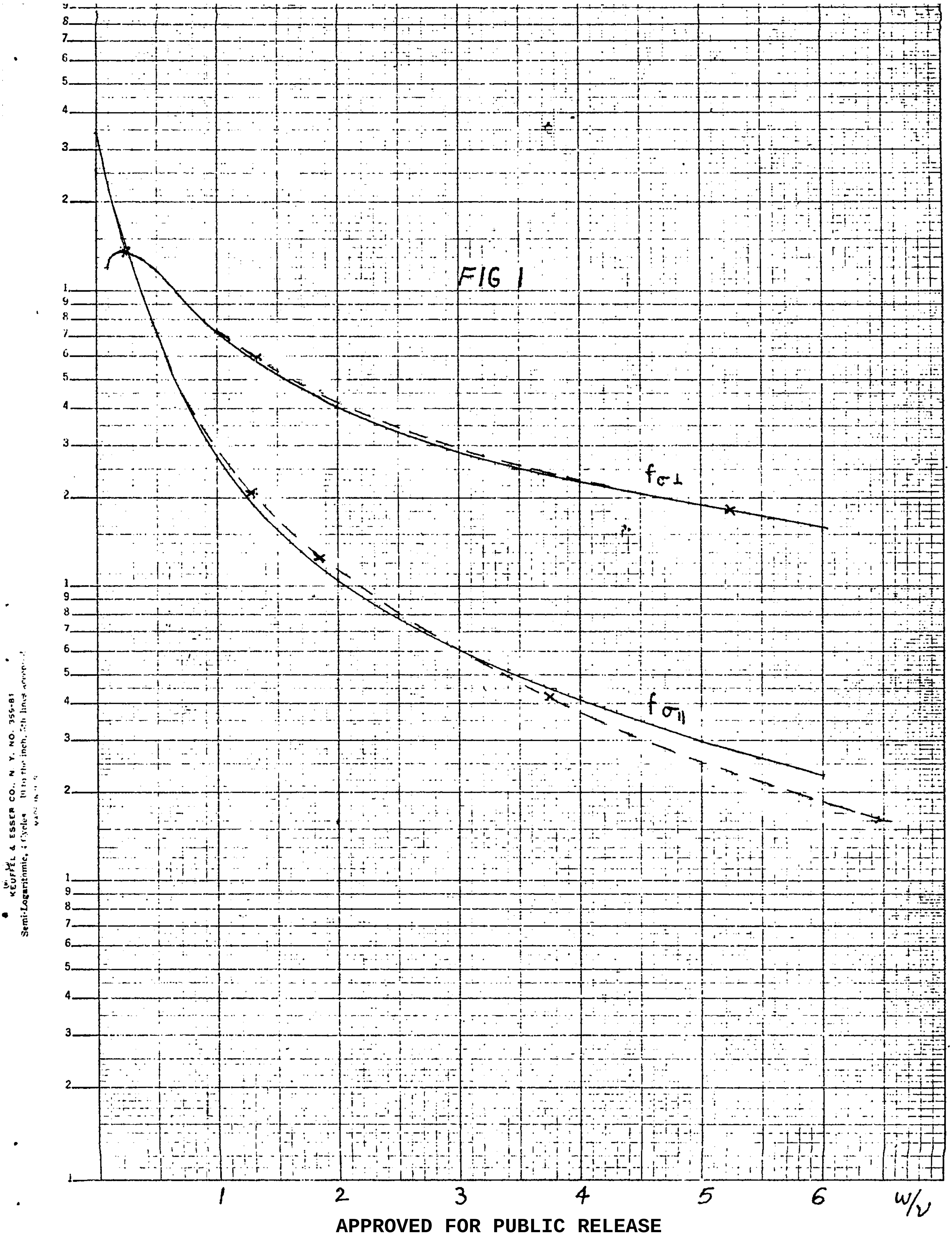
FIG 1
$f_{\sigma\perp}$
$f_{\sigma_{\parallel}}$
1
2
3
4
5
6
w/ν
KEUFFEL & ESSER CO., N. Y. NO. 359-81
Semi-Logarithmic

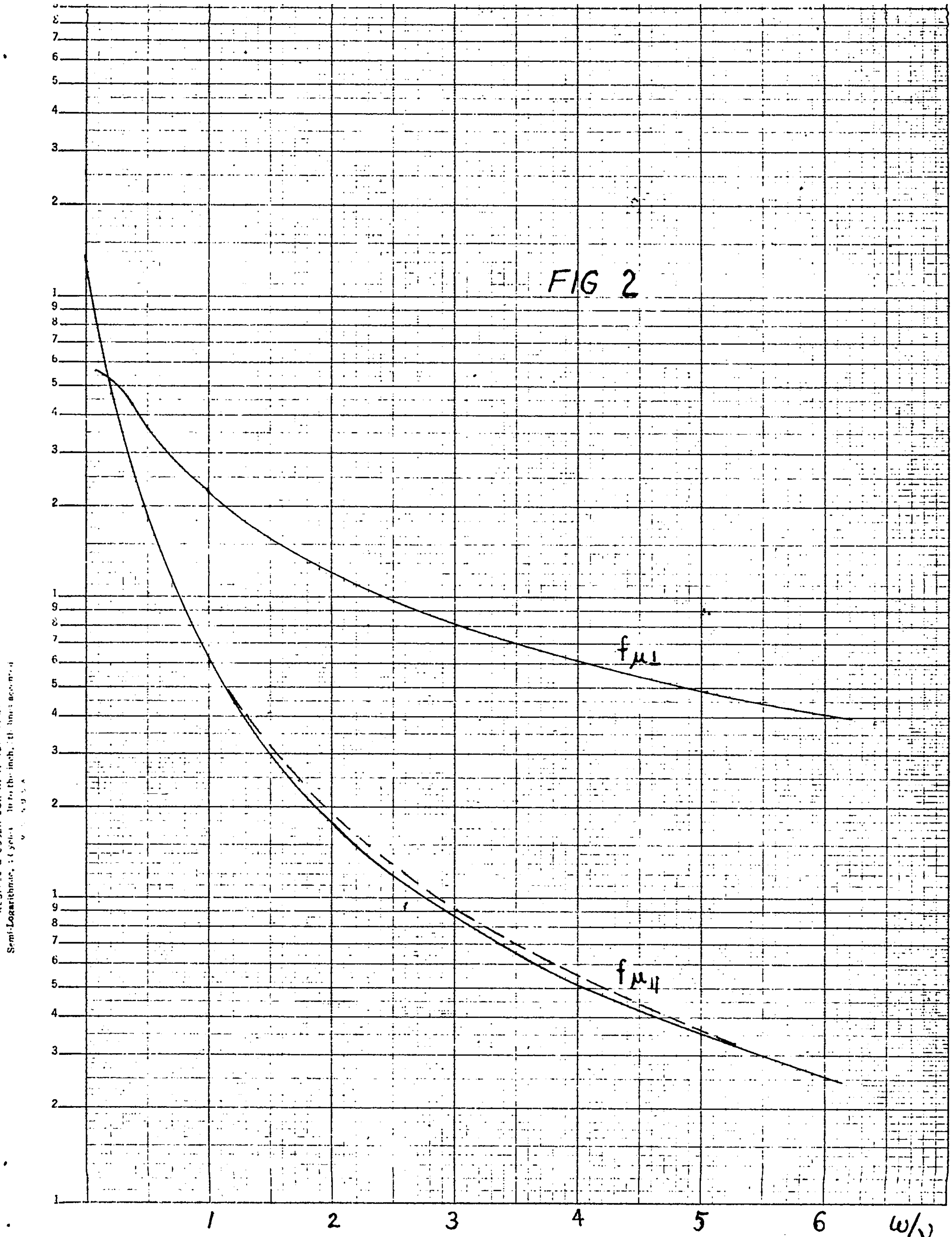

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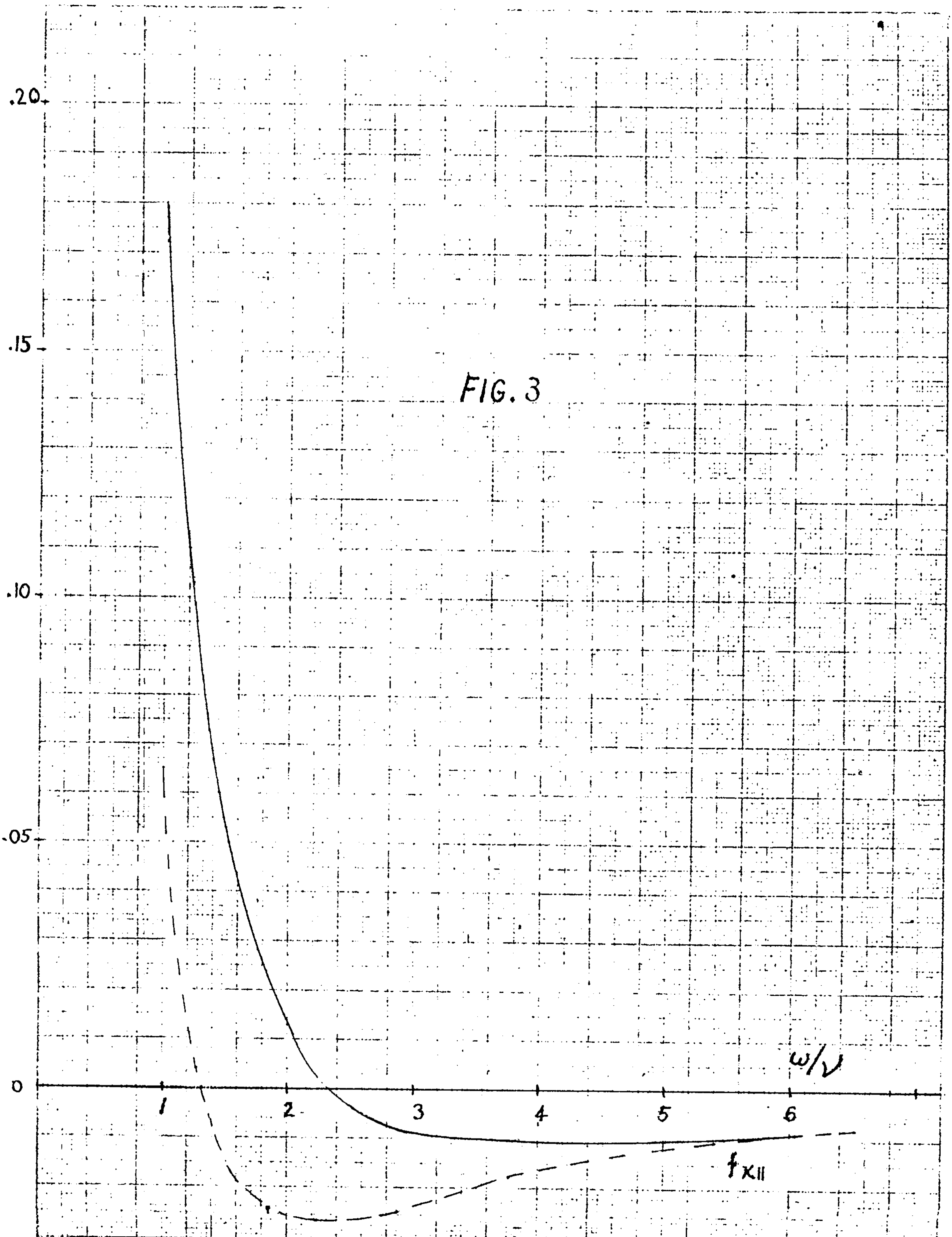
FIG. 3



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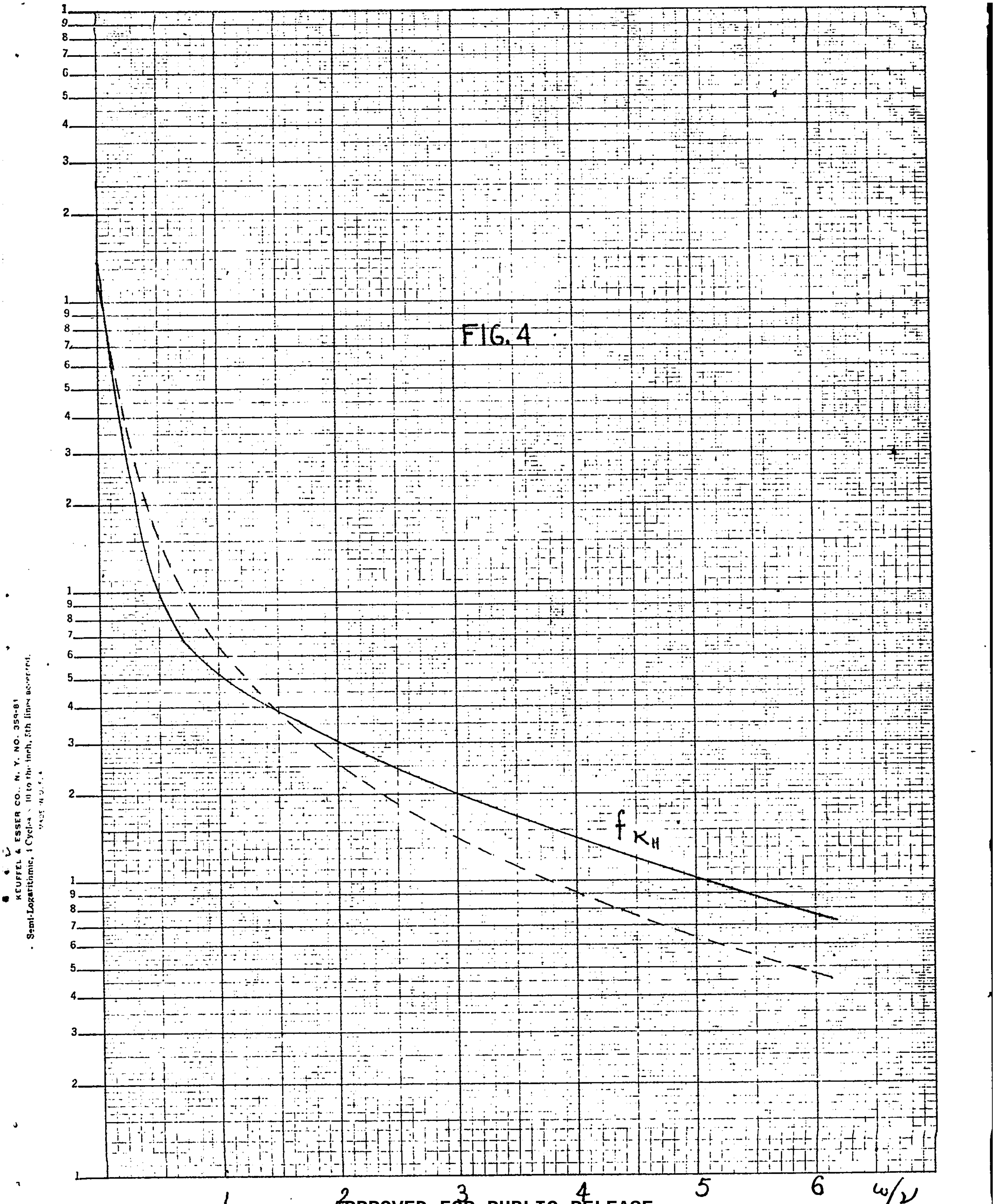

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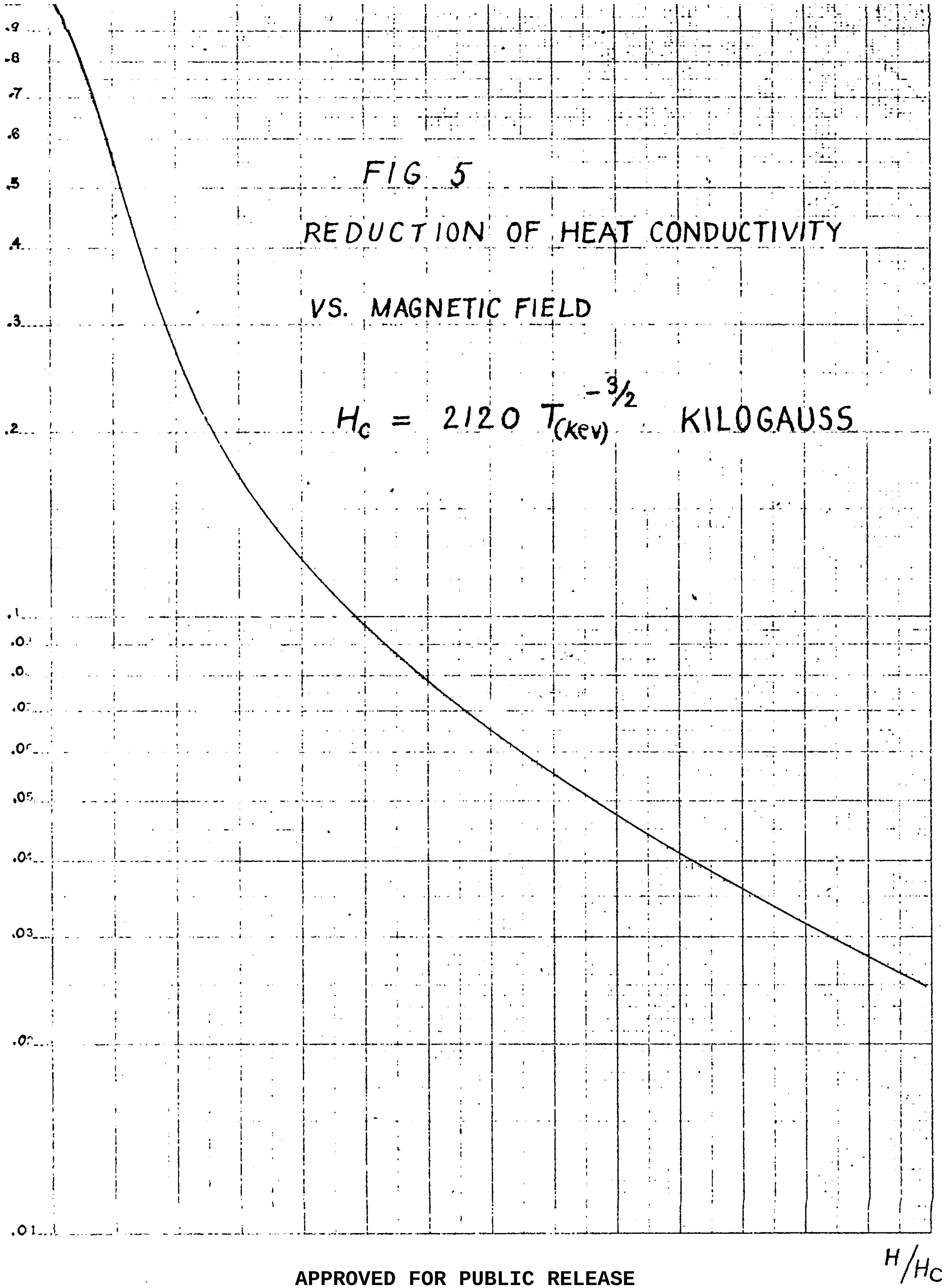
FIG 5
REDUCTION OF HEAT CONDUCTIVITY
VS. MAGNETIC FIELD
$H_c = 2120\ T_{(kev)}^{-3/2}$ KILOGAUSS
.9
.8
.7
.6
.5
.4
.3
.2
.1
.05
.03
.01
H/Hc

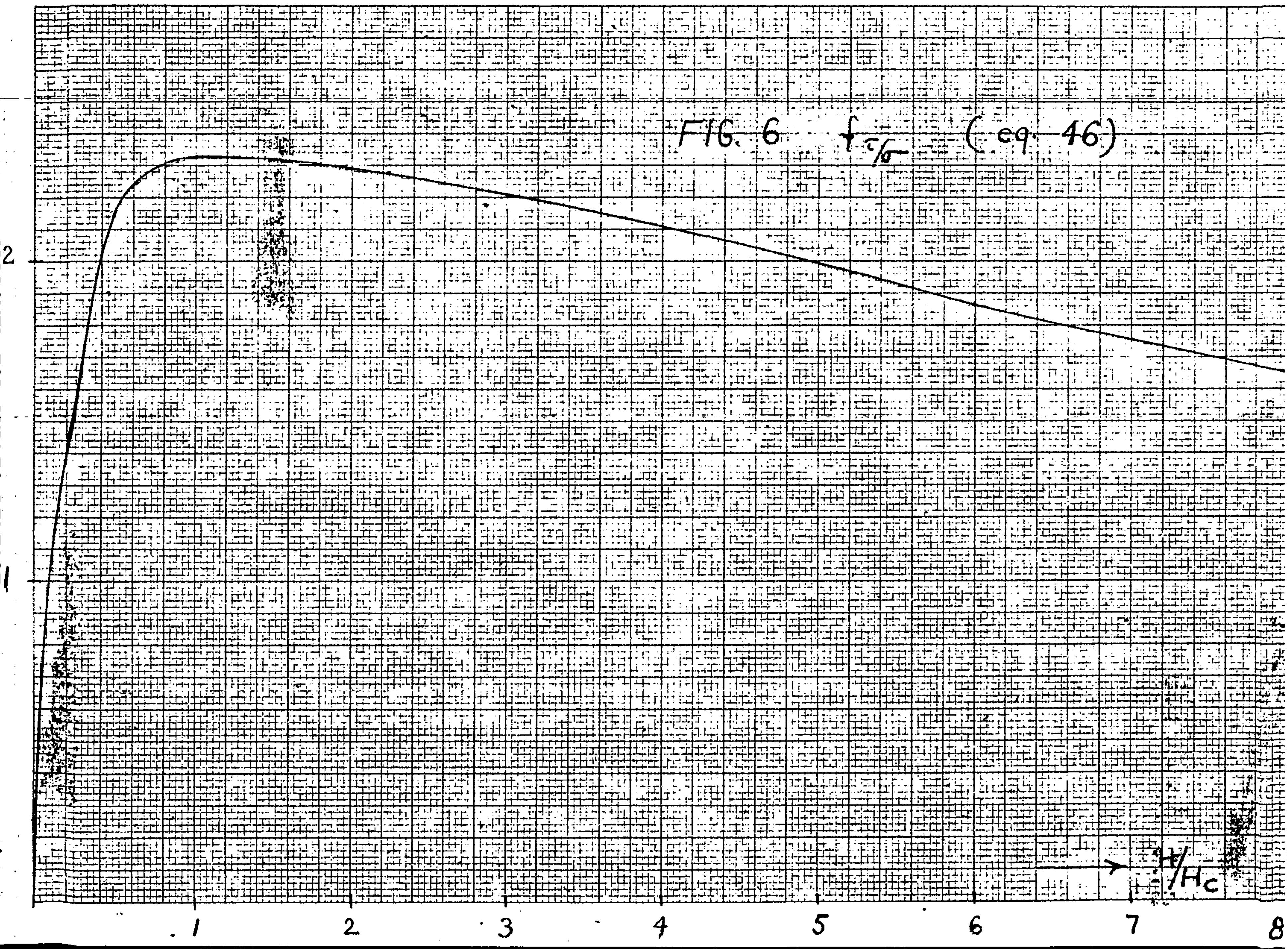

FIG. 6 $f_{\tau/\sigma}$ (eq. 46)

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